Core 4 - January 2008

① a) $\dfrac{3}{9-x^2} = \dfrac{3}{(3-x)(3+x)} = \dfrac{A}{(3-x)} + \dfrac{B}{(3+x)}$

$\rightarrow \quad 3 = A(3+x) + B(3-x)$

$\boxed{x=-3} \quad 3 = B(6) \rightarrow B = \tfrac{1}{2}$

$\boxed{x=3} \quad 3 = A(6) \rightarrow A = \tfrac{1}{2}$

$\therefore \dfrac{3}{9-x^2} = \dfrac{1}{2(3+x)} + \dfrac{1}{2(3-x)} = \tfrac{1}{2}\left(\dfrac{1}{3+x} + \dfrac{1}{3-x}\right)$

b) $\displaystyle\int_1^2 \frac{3}{9-x^2} = \tfrac{1}{2}\int_1^2 \frac{1}{3+x} + \tfrac{1}{2}\int_1^2 \frac{1}{3-x}$

$= \tfrac{1}{2}\left[\ln(3+x)\right]_1^2 + -\tfrac{1}{2}\left[\ln(3-x)\right]_1^2$

$= \tfrac{1}{2}\left[\ln(5) - \ln(4)\right] - \tfrac{1}{2}\left[\ln(1) - \ln(2)\right]$

$= \tfrac{1}{2}\left[\ln\left(\dfrac{5 \times 2}{4 \times 1}\right)\right] = \tfrac{1}{2}\ln(10/4) = \tfrac{1}{2}\ln(5/2)$

② a) i) To test $2x-1 \rightarrow$ sub in $x = \tfrac{1}{2}$

$f(\tfrac{1}{2}) = 2(\tfrac{1}{2})^3 + 3(\tfrac{1}{2})^2 - 18(\tfrac{1}{2}) + 8 = 0 \quad \therefore (2x-1)$ is factor

ii)

$$\begin{array}{r} x^2 + 2x - 8 \\ (2x-1)\overline{)\,2x^3 + 3x^2 - 18x + 8} \\ 2x^3 - x^2 \qquad\qquad \\ 4x^2 - 2x \qquad \\ -16x + 8 \end{array}$$

$\rightarrow f(x) = (2x-1)(x^2+2x-8)$

iii) $\dfrac{4x^2+16x}{2x^3+3x^2-18x+8} = \dfrac{4x(x+4)}{(2x-1)(x^2+2x-8)}$

$= \dfrac{4x(x+4)}{(2x-1)(x+4)(x-2)} = \dfrac{4x}{(2x-1)(x-2)}$

b) $\dfrac{2x^2}{(x+5)(x-3)} = A + \dfrac{B+Cx}{(x+5)(x-3)}$

$2x^2 = A(x+5)(x-3) + B + Cx$

A must = 2 to get $2x^2$

$\boxed{x=0} \quad 0 = 2(5)(-3) + B \rightarrow B = 30$

$\boxed{x=3} \quad 18 = B + C(3) \rightarrow C = -4$

③ a) $(1+x)^{1/2} = 1 + (1/2)x + \frac{(1/2)(-1/2)x^2}{2}$

$= 1 + 1/2\,x - 1/8\,x^2$

b) $(1 + 3/2x)^{1/2} = 1 + 1/2(3/2\,x) - 1/8(3/2\,x)^2$

$= 1 + 3/4\,x - 9/32\,x^2$

c) $\sqrt{\frac{2+3x}{8}} = \sqrt{\frac{1}{4}} \times \sqrt{\frac{2+3x}{2}}$

$= 1/2\,(1 + 3/2\,x)^{1/2}$

$= 1/2\,[1 + 3/4\,x - 9/32\,x^2] = 1/2 + 3/8\,x - 9/64\,x^2$

④ a) i) $t = 0$, $P = 20$ $\rightarrow 20 = A \times 1 \rightarrow A = 20$

ii) $t = 60$, $P = 2000$ $\rightarrow 2000 = 20k^{60}$

$100 = k^{60} \rightarrow k = \sqrt[60]{100}$

~~ln(100) = 60 ln~~ $= 1.079775...$

iii) 1 Jan 2008 $\rightarrow t = 123$

$P = 20 \times 1.079775^{123} = £251\,780.422$

$= £252\,000$ (nearest 1,000)

b) Solve: $15 \times 1.082709^t = 20 \times 1.079775^t$

$\frac{15}{20} = \frac{1.079775^t}{1.082709^t} = \left(\frac{1.079775}{1.082709}\right)^t$

$\ln(15/20) = t \ln\left(\frac{1.079775}{1.082709}\right)$

$\ln(15/20) \div \ln\left(\frac{1.079775}{1.082079}\right) = t = 106.0169...$

$= 1991$ (year)

⑤ a) i) $t = 1/2 \rightarrow x = 2(1/2) + 1/(1/2)^2 = 5$

$\rightarrow y = 2(1/2) - 1/(1/2)^2 = -3$

ii) $x = 2t + 1/t^2$ $\qquad y = 2t - 1/t^2$

$dx/dt = 2 - 2t^{-3}$ $\qquad dy/dt = 2 + 2t^{-3}$

$= 2 - 2/t^3$ $\qquad = 2 + 2/t^3$

$dy/dx = dy/dt \times dt/dx = \frac{(2 + 2/t^3)}{(2 - 2/t^3)}$ ~~(h2 - 2/t~~

when $t = 1/2$

$x = 5$

$y = -3$

$dy/dx = \dfrac{2 + 2/(1/2)^3}{2 - 2/(1/2)^3}$

$= \cancel{\ldots} = \cancel{\ldots}$

$= 18/-14 = -9/7$

$y - y_1 = m(x - x_1)$

$y + 3 = -9/7\,(x - 5)$

b) $(x-y)(x+y)^2 = (2t + 1/t^2 - 2t + 1/t^2)(2t + 1/t^2 + 2t - 1/t^2)^2$

$= 2/t^2\,(4t)^2$

$= 2/t^2\,(16t^2) = 32$

6) $3xy - 2y^2 = 4$

$\rightarrow 3y + 3xy\,dy/dx - 4y\,dy/dx = 0$

$\boxed{3xy}$ $u = 3x$ $v = y$

$du/dx = 3$ $dv/dx = y\,dy/dx$

$\rightarrow 3y + 3xy\,dy/dx$

At (2, 1)

$\rightarrow 3 + 6\,dy/dx - 4\,dy/dx = 0$

$2\,dy/dx = -3$

$dy/dx = -3/2$ = gradient at (2, 1)

7) a) i) $6\sin\theta + 8\cos\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$6 = R\cos\alpha$

$8 = R\sin\alpha$

$\rightarrow 8/6 = \tan\alpha \rightarrow \alpha = 53.130\ldots$

$= 53.1^\circ$

$R = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

$\rightarrow 10\sin(\theta + 53.1^\circ)$

ii) $6\sin(2x) + 8\cos(2x) = 7$

$= 10\sin(\theta + 53.1^\circ) = 7$ where $\theta = 2x$

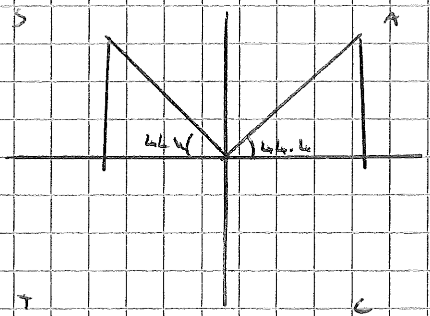
$\rightarrow \sin(\theta + 53.1) = 7/10$

$\cancel{\theta + 53.1 = 44.}$

$\rightarrow \sin(t) = 0.7$ where $t = 2x + 53.1$

$\sin(t) = 0.7$

$\Rightarrow t = 44.42...$



$t = 2x + 53.1$

$0 < x < 360$

$53.1 < t < 773.1$

$t = 135.57, \; 404.42, \; 495.57, \; 764.43$

$x = \dfrac{t - 53.1}{2}$

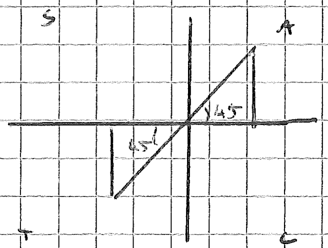
$x = 41.2, \; 175.7, \; 221.2, \; 355.7$

b) i) $\dfrac{\sin(2x)}{1 - \cos(2x)} = \dfrac{2\sin(x)\cos(x)}{1 - (1 - 2\sin^2(x))}$

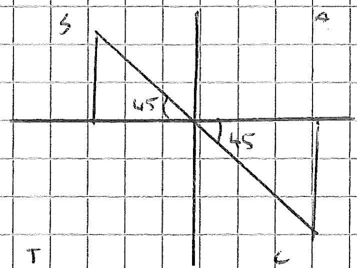
$= \dfrac{2\sin(x)\cos(x)}{2\sin^2(x)} = \dfrac{\cos(x)}{\sin(x)} = \dfrac{1}{\tan(x)}$

ii) $\dfrac{1}{\tan(x)} = \tan(x) \Rightarrow \tan^2(x) = 1$

$\Rightarrow \tan(x) = \pm\sqrt{1} = \pm 1$



$x = 45, \; 225$



$x = 135, \; 315$

(8) $\dfrac{dy}{dx} = \dfrac{3\cos(3x)}{y}$

$\int y \, dy = \int 3\cos(3x) \, dx$

$y^2/2 = \sin(3x) + c$

$y = 2, \; x = \pi/2 \Rightarrow 4/2 = \sin(3\pi/2) + c$

$2 = -1 + c \Rightarrow c = 3$

$\Rightarrow y^2/2 = \sin(3x) + 3$

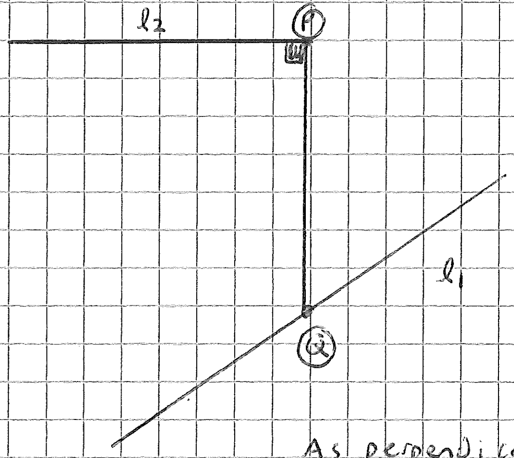
$y^2 = 2\sin(3x) + 6$

⑨ a) i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \\ -4 \\ -3 \end{pmatrix}$

ii) $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$

b) i)
$$\left.\begin{aligned} 1 + \mu &= -2 \\ -3 &= -3 \\ -1 - 2\mu &= 5 \end{aligned}\right\}$$ All satisfied by $\mu = -3$ $\therefore$ P lies on line

ii)



$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 2 + 2 + 2\lambda \\ 3 + 5 - 4\lambda \\ -5 + 1 - 3\lambda \end{pmatrix} = \begin{pmatrix} 4 + 2\lambda \\ 8 - 4\lambda \\ -4 - 3\lambda \end{pmatrix}$

As perpendicular, $a.b = 0$

$\begin{pmatrix} 4 + 2\lambda \\ 8 - 4\lambda \\ -4 - 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 4 + 2\lambda + 8 + 6\lambda = 0$

$12 + 8\lambda = 0$

$\rightarrow \lambda = -12/8 = -1.5$

$\therefore OQ = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} = \begin{matrix} 2 - 1.5(2) \\ 5 - 1.5(-4) \\ 1 - 1.5(-3) \end{matrix} = \begin{pmatrix} -1 \\ 11 \\ 5.5 \end{pmatrix}$

$\therefore Q = (-1, 11, 5.5)$